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Water-wave problems, their mathematical solution and physical interpretation

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Abstract A typical linear water-wave problem is a boundary-value problem involving partial differential equations and boundary conditions. Such a problem is usually solved by applying a sequence of mathematical arguments, and it would be helpful if some or all of the successive steps in this sequence could be given a physical interpretation. In the author's experience this is generally not possible. Illustrative examples are presented.

Keywords Linear · Water · Waves

1 Introduction

The water-wave problems which I have solved during my career have mostly been linear. Typically a physical problem was modelled as a system of differential equations and boundary conditions, to which mathematical techniques were applied until a solution was obtained. My first problem was in oceanography, during World War II: How do ocean waves propagate from a storm centre? As a mathematical model for the storm we considered an instantaneous localized disturbance acting on the free surface of a frictionless fluid, and so we came to study the classical Cauchy–Poisson problem (1815); see [1, Art. 255]. Its solution at distance r at time t takes the form of an integral. This simplifies when the asymptotic parameter gt^2/r is large; each wave frequency then propagates from the centre with the appropriate group velocity which is equal to half the phase velocity. When the disturbance is extended in space and time, a wave spectrum is generated which varies with time. An observer on a distant shore can measure the wave spectrum at a given time, and by using the theoretical group velocity can trace each observed frequency back in space and time. It was found that these frequencies could be traced back into storm areas. In this problem it was the mathematical treatment that suggested a physical notion, the group velocity, and this gave us a physical insight into our problem. It is reasonable to hope that there are many problems for which we may gain physical insights by studying their mathematical solutions. However, in reviewing my own work I find that I

To Nick Newman on his seventieth birthday

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have often not been able to attach a physical meaning to the most essential parts of the mathematical argument. On the contrary, almost every solution has required some special mathematical trick which appears remote from the physical aspects of the problem, in many cases an asymptotic trick. I regard this as a kind of paradox, and I drew attention to it in my Weinblum Lecture in 1985 [2]. In the present paper I shall give additional illustrations by re-examining the Kelvin shipwave pattern, then pass on to several problems involving the half-immersed circle of radius a oscillating vertically with small amplitude, and conclude with the concentrated localized disturbance (the Cauchy–Poisson problem) on finite constant depth.

2 Kelvin's shipwave pattern

We take polar coordinates travelling with a concentrated pressure point moving at constant velocity U, and write $N = gr/U^2 = Kr$. It can be shown by using a Fourier technique that the wave contribution to the elevation at the point (r, θ) is given by the imaginary part of

$$f_*(N,\theta) = \int_{\infty\exp(-7\pi \mathrm{i}/8)}^{\infty\exp(\pi \mathrm{i}/8)} (1+u^2)\exp\{\mathrm{i}N\Phi(u,\theta)\}\mathrm{d}u, \tag{2.1}$$

where

$$\Phi(u,\theta) = (\cos\theta - u\sin\theta)(1+u^2)^{1/2}, \tag{2.2}$$

and where it is assumed that the path of integration passes between the two branch-points $u = \pm\mathrm{i}$ (see e.g. [3]). This simplifies when N is large; we use an asymptotic technique, the Method of Stationary Phase in its complex-variable form the Method of Steepest Descents, which shows that the leading contributions for large N come from points u where the function Φ is stationary, i.e., from points u at which

$$\frac{\mathrm{d}}{\mathrm{d}u}\left((\cos\theta - u\sin\theta)(1+u^2)^{1/2}\right) = 0, \tag{2.3}$$

i.e., from the two points

$$u_\pm(\theta) = \frac{1}{4}\{\cot\theta \pm (\cot^2\theta - 8)^{1/2}\},$$

the relevant points of stationary phase. Near the points $u = u_\pm(\theta)$ the function Φ varies as $[u - u_\pm(\theta)]^2$, and the surface elevation therefore consists of two wave systems

$$\zeta_*(N,\theta) = \frac{F_1(\theta)}{N^{1/2}}\cos(Nf_1(\theta)+\epsilon_1) + \frac{F_2(\theta)}{N^{1/2}}\cos(Nf_2(\theta)+\epsilon_2), \tag{2.4}$$

known as the diverging and transverse systems; see [1, Art. 256]. However, according to this approximation the amplitude of each system becomes infinite when one of the critical lines $\theta = \pm\theta_c$ is approached, where $\cot\theta_c = 2^{3/2}$, although the original integral (2.1) was finite; this failure of the method in its simplest form occurs whenever two points of stationary phase approach each other. The wave amplitude on each critical line was studied in 1908 by Havelock, see [4], who found that $\Phi(u,\theta_c)$ near $u = u(\theta_c)$ varies as $[u - u(\theta_c)]^3$, and that the amplitude along a critical line is finite and varies as $N^{-1/3}$, by a straightforward extension of the original stationary-phase argument. We wish to know more: what is the behaviour for large N of the pattern near (but not on) the critical line? The problem of two nearly coincident points of stationary phase had been studied many years earlier by G.B. Airy in his treatment of caustics, in which he introduced Airy functions, given in modern notation by

$$\begin{aligned}\mathrm{Ai}(X) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos\left(\frac{1}{3}t^3 + Xt\right)\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{\infty\exp(5\pi \mathrm{i}/6)}^{\infty\exp(\pi \mathrm{i}/6)}\exp\left(\frac{1}{3}it^3 + \mathrm{i}Xt\right)\mathrm{d}t,\end{aligned} \tag{2.5}$$

involving a cubic polynomial in the exponent as would be expected. Treatments of our problem involving Airy functions have been given; they use various cubic approximations to $\Phi(u,\theta)$ near $u = u(\theta_c)$, and typically lead to products like

$$\frac{1}{N^{1/3}}\mathrm{Ai}\left\{\frac{3}{2^{1/2}}N^{2/3}(\theta-\theta_c)\right\}\sin\left\{N\left(-\frac{3^{1/2}}{2}+\frac{3^{1/2}}{2^{1/2}}(\theta-\theta_c)\right)\right\}, \tag{2.6}$$

where the Airy function depends on a single parameter which is the product of a large factor $N^{2/3}$ and a small factor. (The precise form of the small factor depends on the cubic approximation.) It can be shown that none of these results joins smoothly with the earlier approximation (2.4) inside the critical angle; in fact, these results are only valid in narrow zones of the form $|\theta - \theta_c| < AN^{-p}$ near the critical line. This is not satisfactory. The best-known approach is due to Hogner, who used different cubic approximations on the two sides of the critical line; his result has a similar restricted region of validity and also an apparent discontinuity when the critical line is crossed.

2.1 Change of variable

How can we obtain a uniform approximation for an integral involving two points of stationary phase that tend to coincide when a parameter (here the angle θ) tends to a critical value θ_c? Here is the mathematical trick: following Chester et al. [5] we look for a new variable of integration v such that the phase function takes the form of an exact cubic in v:

$$\Phi(u,\theta) = -\frac{1}{3}v^3 + \mu(\theta)v - \nu(\theta). \tag{2.7}$$

This is not just a simple mapping from u to v, for near $u = u_c$ we expect (2.7) to be a (3,3) transformation: to each value of u there correspond three values of v, and to each value of v there correspond three values of u. For every pair of corresponding values $u \leftrightarrow v$ we have locally

$$\frac{\partial\Phi(u,\theta)}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}v} = -v^2 + \mu(\theta). \tag{2.8}$$

If the transformation from u to v is to contain a regular mapping in an interval including the two zeros $u = u_\pm(\theta)$, then the zeros $u = u_\pm(\theta)$ on the left-hand side of (2.8) must correspond to the zeros $v = \pm\mu^{1/2}(\theta)$ on the right-hand side, and so we obtain the two equations

$$\Phi(u_\pm(\theta),\theta) = \pm\frac{2}{3}\mu^{2/3}(\theta) - \nu(\theta) \tag{2.9}$$

for $\mu(\theta)$ and $\nu(\theta)$ in (2.7). It follows that $\mu(\theta)$ and $\nu(\theta)$ are analytic near $\theta = \theta_c$,

$$\mu(\theta) = -\frac{3}{2^{1/2}}(\theta-\theta_c) + O((\theta-\theta_c)^2),$$
$$\nu(\theta) = -\frac{3^{1/2}}{2} + \frac{3^{1/2}}{2^{1/2}}(\theta-\theta_c) + O((\theta-\theta_c)^2).$$

The conditions (2.9) are necessary for regularity, and it was shown by Chester et al. [5] that they are also sufficient: the (3,3) transformation (2.7) breaks up locally into a regular (1,1) transformation and a (2,2) transformation. The method was applied to the Kelvin pattern in [3] and it was found that the expansion resulting from the (1,1) branch is valid in a finite angle and agrees with the steepest-descents expansion (2.4) when $N^{2/3}|\theta_c - \theta|$ is large. (Near the track $\theta = 0$ of the disturbance a modification is needed [6].)

Comment on 2.1 change of variable: the relation (2.7) is implicit and multi-valued, and therefore nonphysical.
We still need to determine the form of the expansion and to show that it is indeed asymptotic. We write

$$f_*(N,\theta) = \int (1+u^2)\frac{\mathrm{d}u}{\mathrm{d}v}\exp\{\mathrm{i}N\Phi(u,\theta)\}\mathrm{d}v$$
$$= \int G_0(v,\theta)\exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v-\nu(\theta)\right)\right\}\mathrm{d}v, \text{say.} \tag{2.10}$$

For the moment, let us assume formally that the change of variable is analytic not only locally but for all relevant values of u. We can then obtain the complete expansion by constructing a Bleistein sequence. We write

$$G_0(v,\theta) = a_0(\theta) + b_0(\theta)v + (v^2-\mu)H_0(v,\theta) \tag{2.11}$$

from which it follows that $a_0(\theta)$ and $b_0(\theta)$ satisfy

$$G_0(\pm\mu^{1/2},\theta) = a_0(\theta) \pm b_0(\theta)\mu^{1/2}.$$

The leading terms in the elevation thus contain the expression

$$2\pi\exp(-\mathrm{i}\nu(\theta))\left(\frac{a_0(\theta)}{N^{1/3}}\mathrm{Ai}(-N^{2/3}\mu(\theta)) + \frac{b_0(\theta)}{N^{2/3}}\mathrm{Ai}'(-N^{2/3}\mu(\theta))\right), \tag{2.12}$$

where in the Airy function the argument $-N^{2/3}\mu(\theta)$ is once again the product of a large factor and a small factor. It will next be shown that the complete asymptotic expansion has the form

$$\frac{2\pi}{N^{1/3}}\exp(-\mathrm{i}\nu(\theta))\mathrm{Ai}(-N^{2/3}\mu(\theta))\left(a_0(\theta)+\frac{a_1(\theta)}{(\mathrm{i}N)}+\frac{a_2(\theta)}{(\mathrm{i}N)^2}+\cdots\right)$$
$$+\frac{2\pi}{N^{2/3}}\exp(-\mathrm{i}\nu(\theta))\mathrm{Ai}'(-N^{2/3}\mu(\theta))\left(b_0(\theta)+\frac{b_1(\theta)}{(\mathrm{i}N)}+\frac{b_2(\theta)}{(\mathrm{i}N)^2}+\cdots\right). \tag{2.13}$$

To obtain the higher terms we note that

$$\int G_0(v,\theta)\exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v-\nu(\theta)\right)\right\}\mathrm{d}v$$
$$-a_0(\theta)\int\exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v-\nu(\theta)\right)\right\}\mathrm{d}v$$
$$-b_0(\theta)\int v\exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v-\nu(\theta)\right)\right\}\mathrm{d}v$$
$$=\int(v^2-\mu)H_0(v,\theta)\exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v-\nu(\theta)\right)\right\}\mathrm{d}v$$
$$=-\frac{1}{\mathrm{i}N}\int H_0(v,\theta)\frac{\mathrm{d}}{\mathrm{d}v}\exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v-\nu(\theta)\right)\right\}\mathrm{d}v$$
$$=\frac{1}{\mathrm{i}N}\int\left(\frac{\partial}{\partial v}H_0(v,\theta)\right)\exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v-\nu(\theta)\right)\right\}\mathrm{d}v$$
by integration by parts
$$=\frac{1}{\mathrm{i}N}\int G_1(v,\theta)\exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v-\nu(\theta)\right)\right\}\mathrm{d}v, \text{ say.}$$

As in (2.11) we can now write

$$G_1(v,\theta) = a_1(\theta) + b_1(\theta)v + (v^2-\mu)H_1(v,\theta), \tag{2.14}$$

and repeat our procedure to obtain the higher terms.

2.2 Asymptotic bounds

To prove that the resulting series (2.13) is asymptotic we need to show that the remainder term at any stage is of the same order as the next contribution to the series, i.e., that

$$\left|\int G_p(v,\theta)\exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v\right)\right\}\mathrm{d}v\right| \\ \le M_p\left|\int \exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v\right)\right\}\mathrm{d}v\right| + M_p\left|\int v\exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v\right)\right\}\mathrm{d}v.\right| \tag{2.15}$$

The terms on the right-hand side are multiples of the Airy function and its derivative and take different forms, according as $N^{2/3}\mu(\theta)$ is large negative, moderate negative, small negative, small positive, moderate positive or moderate large, and the same holds for the term on the left-hand side. The difficult comparisons in these various zones can be avoided by another trick: since we have formally assumed that $G_0(v,\theta)$ is analytic everywhere, we can now define the functions $A(\theta)$ and $B(\theta)$ as the solution of the equations (see (2.13) above)

$$\int_{C(1)} G_0(v,\theta)\exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v\right)\right\}\mathrm{d}v \\ = A(\theta)\int_{C(1)} \exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v\right)\right\}\mathrm{d}v + B(\theta)\int_{C(1)} v\exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v\right)\right\}\mathrm{d}v, \tag{2.16}$$

$$\int_{C(2)} G_0(v,\theta)\exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v\right)\right\}\mathrm{d}v \\ = A(\theta)\int_{C(2)} \exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v\right)\right\}\mathrm{d}v + B(\theta)\int_{C(2)} v\exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v\right)\right\}\mathrm{d}v, \tag{2.17}$$

where $C(1)$ extends from $\infty\exp(\pi\mathrm{i})$ to $\infty\exp(\pi\mathrm{i}/3)$ and $C(2)$ extends from $\infty\exp(\pi\mathrm{i}/3)$ to $\infty\exp(-\pi\mathrm{i}/3)$. Then it follows by addition that

$$\int_{C(3)} G_0(v,\theta)\exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v\right)\right\}\mathrm{d}v \\ = A(\theta)\int_{C(3)} \exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v\right)\right\}\mathrm{d}v + B(\theta)\int_{C(3)} v\exp\left\{\mathrm{i}N\left(-\frac{1}{3}v^3+\mu(\theta)v\right)\right\}\mathrm{d}v, \tag{2.18}$$

where $C(3)$ extends from $\infty\exp(-\pi\mathrm{i}/3)$ to $\infty\exp(\pi\mathrm{i})$. It can now be shown by analytic continuation of the integrals that $A(\theta)$ and $B(\theta)$ are regular analytic functions in a fixed circle $|\theta-\theta_c|\le\rho$. (For each value of θ the appropriate pair of contours must be chosen. We are using the property that $\mu(\theta)$ is dominated by a linear term near $\theta=\theta_c$.) By the maximum-modulus principle the functions $|A(\theta)|$ and $|B(\theta)|$ inside the circle are therefore bounded by their maximum values on the circle $|\theta-\theta_c|=\rho$. These values are readily found from nonuniform asymptotics, because on the circle the two points of stationary phase are well separated. The required bound (2.15) for the remainder is thus obtained. If $G_0(v,\theta)$ is not everywhere analytic the argument remains essentially valid ; for the required modifications see Ursell [7, 8].

Comment on 2.2: asymptotic bounds: The maximum-modulus theorem has no obvious physical connection with the problem. We also note that the argument is counter-intuitive: we are interested in the bound for small $|\mu(\theta)|$ but the proof uses values of the integrals for which $|\mu(\theta)|$ is not small.

3 The heaving semi-circular cylinder, periodic motion, infinite depth of fluid

We write $x = r\sin\theta, y = r\cos\theta$. The equations are

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x,y) = 0 \text{ when } r > a, \quad y > 0, \tag{3.1}$$

$$K\phi + \frac{\partial \phi}{\partial y} = 0 \text{ when } |x| > a, \quad y = 0, \tag{3.2}$$

$$\frac{\partial \phi}{\partial r} = V_0 \cos\theta \text{ when } r = a, \quad -\pi/2 \le \theta \le \pi/2. \tag{3.3}$$

Also, at infinity the waves travel outwards. The text-book approach to a problem of this type is through integral equations and is due to Fredholm (1900). This is described below; it was not known to me when I began to work on this problem in 1946.

3.1 Form of the expansion

Instead, I wrote the potential as the sum of a wave source at the origin, together with an infinite set of wavefree potentials:

$$\begin{aligned}\phi(x,y) &= \phi_*(r,\theta) \\ &= A_0 \int_0^\infty \exp(-kr\cos\theta)\cos(kr\sin\theta)\frac{\mathrm{d}k}{k-K} \\ &\quad + \sum_{m=1}^{\infty} A_m a^{2m}\left(\frac{\cos 2m\theta}{r^{2m}} + \frac{K}{2m-1}\frac{\cos(2m-1)\theta}{r^{2m-1}}\right),\end{aligned} \tag{3.4}$$

where the contour of integration in the k-plane passes below the pole $k = K$. How can this expansion be justified? By another trick (see also [9]): define the harmonic function

$$\Phi(x,y) = \left(K + \frac{\partial}{\partial y}\right)\phi(x,y), \tag{3.5}$$

then $\Phi(x,0) = 0$ when $|x| > a$, and it follows that $\Phi(x,y)$ can be continued analytically into the whole domain $r > a$ by means of the relation $\Phi(x,y) = -\Phi(x,-y)$, also Φ tends to 0 at infinity. It follows that there is an expansion

$$\Phi(x,y) = \Phi_*(r,\theta) = \sum_{m=0}^{\infty} B_m a^{2m}\frac{\cos(2m+1)\theta}{r^{2m+1}}. \tag{3.6}$$

It is easy to verify that Eq. 3.5 has the solution

$$\begin{aligned}\phi(x,y) &= \phi_*(r,\theta) \\ &= A_0\int_0^\infty \mathrm{e}^{-ky}\cos kx\frac{\mathrm{d}k}{k-K} + \sum_{m=1}^{\infty} A_m a^{2m}\left(\frac{\cos 2m\theta}{r^{2m}} + \frac{K}{2m-1}\frac{\cos(2m-1)\theta}{r^{2m-1}}\right) \\ &\quad + C\exp(-Ky)\cos Kx,\end{aligned} \tag{3.7}$$

where

$$A_M \frac{(Ka)^{2M-1}}{(2M-1)!} = -\sum_{m=M}^{\infty} B_m \frac{(Ka)^{2m-1}}{(2m)!} \quad \text{when } M \ge 1, \tag{3.8}$$

$$A_0 = (Ka)^2 A_1 - B_0, \tag{3.9}$$

where the contour of integration passes below the pole $k = K$, and where the constant C in (3.7) is arbitrary. From the radiation condition at infinity we find that $C = 0$.

Comment on 3.1: form of the expansion: Any analytic continuation into a space not covered by fluid is necessarily nonphysical. (This remark also applies to the classical Method of Images.) For alternative methods of analytic continuation see [9].

For every value of Ka the boundary condition (3.3) on the semi-circle now leads to an infinite set of equations in infinitely many unknowns, and the resulting solution can be shown to exist and to have all the required properties; see my Weinblum Lecture [2].

A digression : almost all the coefficients in the infinite set of equations were explicit, and the numerical solution (including added mass and damping) for a range of values of Ka was actually computed for me in 1947 on an electromechanical computing machine at the National Physical Laboratory. The numerical solution by integral equations could not be found until much later.

The series of wavefree potentials converges rapidly for small Ka, but convergence for large Ka is poor, although for infinite Ka the limiting potential might be expected to be the dipole potential

$$\phi_*^{(\infty)}(r,\theta) = -V_0\,\frac{a^2\cos\theta}{r}, \tag{3.10}$$

satisfying the limiting boundary condition $\phi = 0$ on $y = 0$. How can we find the added mass and damping for large Ka? The series of wavefree potentials does not work, for reasons stated in my Weinblum Lecture.

Fritz John's famous papers on water waves appeared in 1949 and 1950; following Fredholm he used an integral equation for the source strength. For our problem the velocity potential would be expressed as a distribution of oscillating wave sources of unknown source strength over the immersed semi-circle. (There is something strange about this approach, for Green's Representation suggests that we should use both sources and dipoles.) The boundary condition on the semi-circle then leads to an integral equation of the second kind for the source strength, and the solution exists except for an enumerable set of *irregular wavenumbers* Ka tending to infinity. Thus, this approach, like mine, is unsuitable for large Ka. We note that these sources generate not only the required physical motion outside the semi-circle but also a wave motion inside the semi-circle. (Physically there is no wave motion inside the solid semi-circle.) It can be seen that the irregular wavenumbers Ka are wavenumbers at which the interior unphysical wave motion has a certain type of resonance.

3.2 Infinite depth, large Ka

To treat large Ka, I placed a wave source of suitable strength at the centre, which stopped the interior unphysical waves from travelling across the cylinder and eliminated the higher irregular wavenumbers, but it did more, the kernel of the integral equation became small for large Ka, see [10]. The equation could then be solved by iteration (for the semi-circle only), the leading asymptotics could then be found, including the force coefficient expressing added mass and damping, see (5.9) below. This solution is also discussed briefly in my Weinblum Lecture and in full detail in [10]. I remember that Sir Thomas Havelock asked me: What about a semi-ellipse, what about finite depth? For these the kernel would tend to a nonzero limiting kernel, and iteration would still be feasible in theory but not in practice.

Comment on 3.2: infinite depth, large Ka: Clearly the modification to the interior unphysical motion is unphysical.

4 The heaving semi-circular cylinder, periodic motion, finite depth of fluid

In due course I suggested this problem (actually the semi-circle on fluid of finite constant depth) to a research student, Philip Rhodes-Robinson. His trick was, to use Green's Theorem with a nonlinear integrand to find the leading correction term in the added mass; it is

$$-\frac{1}{K}\int_a^\infty \left(\frac{\partial\phi_{0h}}{\partial y}\right)^2 \mathrm{d}x, \tag{4.1}$$

where $\phi_{0h}(x,y)$ denotes the potential of the cylinder heaving at infinite frequency on finite depth h: this potential can be found by solving an infinite system of equations with known coefficients and may be regarded as known. (He also found the wavemaking coefficient, but this is not so difficult.) His method was original, he observed that the force on the semi-circular boundary C involves the integral

$$\int_C \phi\frac{\partial y}{\partial n}\,\mathrm{d}s = \frac{1}{V}\int_C \phi\frac{\partial \phi}{\partial n}\,\mathrm{d}s = \frac{1}{V}\int_C \phi\frac{\partial \phi_{0h}}{\partial n}\,\mathrm{d}s,$$

where $\phi_{0h}(x,y)$ is again the potential corresponding to infinite frequency and finite depth h. If we define the two harmonic wave functions

$$\Phi_1(x,y) = \phi_{0h} - \frac{1}{K}\frac{\partial\phi_{0h}}{\partial y}, \quad \text{and} \quad \Phi_2(x,y) = \phi(x,y) - \Phi_1(x,y),$$

it follows that

$$\int\left(\Phi_1\frac{\partial\Phi_2}{\partial n} - \Phi_2\frac{\partial\Phi_1}{\partial n}\right)\mathrm{d}s = 0, \tag{4.2}$$

where the line integral is taken along the boundary of the fluid. There is no contribution from $y = 0$, because both Φ_1 and Φ_2 satisfy the free-surface condition. It can then be shown that to leading order

$$\int_C\left(\Phi_1\frac{\partial\Phi_2}{\partial n} - \Phi_2\frac{\partial\Phi_1}{\partial n}\right)\mathrm{d}s = 0, \tag{4.3}$$

and it can further be shown that the result (4.1) follows. (Rhodes-Robinson's argument was slightly different; see [12].)

Comment on 4: finite depth: I can see no physical reason why an argument of this type should have produced a useful result.

5 The heaving semi-circular cylinder, transient motion, infinite depth of fluid

The velocity potential of the fluid is denoted by $\phi(x,y;t)$ and the vertical displacement of the centre is denoted by $y_0(t)$. The equation of continuity for the fluid is (see [11])

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x,y;t) = 0 \quad \text{when } r > a \text{ and } y > 0. \tag{5.1}$$

The linearized condition of constant pressure at the free surface is

$$\frac{\partial^2\phi}{\partial t^2} - g\frac{\partial\phi}{\partial y} = 0 \quad \text{when} \quad y = 0,\ |x| > a. \tag{5.2}$$

On the semi-circle the radial velocity components of the fluid and the body are equal,

$$\partial\phi/\partial r = \dot{y}_0(t)\ \cos\theta \quad \text{when } r = a,\ -\pi/2 \le \theta \le \pi/2\,. \tag{5.3}$$

The equation of motion of the body is

$$\frac{1}{2}\pi\rho a^2\ddot{y}_0(t) = -2\rho g a y(t) + \rho a\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos\theta[\partial\phi(a\sin\theta, a\cos\theta;t)/\partial t]\,\mathrm{d}\theta + f_0(t), \tag{5.4}$$

where $f_0(t)$ is the applied vertical force. This system is treated by applying the Laplace transform (the standard method for transient problems) in the form of the one-sided Fourier transform

$$\Phi(x,y;\omega) = \int_0^\infty \mathrm{e}^{\mathrm{i}\omega t}\phi(x,y;t)\mathrm{d}t, \tag{5.5}$$

$$Y_0(\omega) = \int_0^\infty \mathrm{e}^{\mathrm{i}\omega t}y_0(t)\mathrm{d}t. \tag{5.6}$$

It is then found, as expected, that $\Phi(x, y; \omega)$ satisfies the same equations as the fluid potential due to the forced heaving of the cylinder with wavenumber $\omega^2 a/g$, which we studied earlier and which may be regarded as known. In particular, we know the force coefficient $\Lambda\{\omega(a/g)^{1/2}\}$ expressing added mass and damping. By taking the inverse Laplace transform, we find that

$$y_0(t) = \frac{1}{2\rho g^{1/2} a^{3/2}} \int_0^t f_0(t')\, h_1\left((t-t')\left(\frac{g}{a}\right)^{1/2}\right) \mathrm{d}t', \tag{5.7}$$

where

$$h_1(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}u\tau}}{1 - \frac{1}{4}\pi u^2 (1 + \Lambda(u))} \,\mathrm{d}u. \tag{5.8}$$

Thus, Eq. 5.8 represents the transient motion of the cylinder due to an impulsive force. Similarly we can find the transient motion from rest due to an initial vertical displacement. In principle we can now proceed to find the motion of the fluid but this has not been done.

There remains the problem of computing (5.8) numerically, which was solved by Maskell [13]. This integral converges only slowly, for we know from Sect. 3 that for large real u (positive and negative) we have

$$\Lambda(u) \sim 1 - \frac{4}{3\pi u^2} + \cdots . \tag{5.9}$$

Evidently the convergence of (5.8) would be much improved if we could deform the contour of integration into the lower half u-plane where the exponential factor is exponentially small for large τ, for instance by taking a contour passing above $u = 0$ from $\infty \exp(5\pi \mathrm{i}/4)$ to $\infty \exp(-\pi \mathrm{i}/4)$. We have shown in Sect. 3 above how (5.9) can be proved for real u, and it is not difficult to see that the proof can be extended to the whole of the upper half u-plane. The extension to any part of the lower half u-plane was found to be more difficult, our trick was, instead of a central wave source to use a central wave singularity of order approximately equal to $Ka/2$. The convergence of (5.8) for small times τ could also be improved, by subtracting known integrals. The motion is found to be a damped harmonic motion arising from the pole of the integrand, together with a term of simple shape arising from the contour integral.

Let us next consider briefly the transient motion for a semi-circle on finite depth, or a semi-ellipse on infinite depth. We would need the analytic continuation of $\Lambda(u)$ into the lower half u-plane, using perhaps an extension of Rhodes-Robinson's argument. Or perhaps a weaker result would be sufficient. This problem is still unsolved.

Comment on 5: transient motion: The mathematical arguments given above have no obvious physical meaning.

6 Water waves on finite depth due to an impulse

Nick Newman has been a great influence in my life since I first met him in 1957, but we never formally collaborated until nearly 40 years later, on a problem which well illustrates the difficulties which I have described. We had a third collaborator, J.-M. Clarisse. Our problem was the three-dimensional Cauchy–Poisson problem: to find the waves due to an instantaneous localized impulse or impulsive elevation on the free surface of water of finite depth h [14]. Difficulties arise at the wavefront where r/t is nearly equal to the maximum group velocity $(gh)^{1/2}$, and this was the region in which we were interested. We now re-normalize length and time so that $g = 1, h = 1$.

In the corresponding two-dimensional problem there are two nearly coincident saddle-points, and the solution involves the Airy function $\mathrm{Ai}(-2^{1/3} t^{2/3} \epsilon)$ and its derivative, where

$$\frac{2^{3/2}}{3} \epsilon^{3/2} = (k_0 \tanh k_0)^{1/2} \frac{\sinh k_0 \cosh k_0 - k_0}{2 \sinh k_0 \cosh k_0} = \frac{1}{3} k_0^3 + \mathrm{O}(k_0^5),$$

and k_0 is the root of

$$\frac{x}{t} = \left(\frac{\tanh k_0}{k_0}\right)^{1/2} \frac{\sinh k_0 \cosh k_0 + k_0}{2\sinh k_0 \cosh k_0} = 1 - \frac{1}{2}k_0^2 + \mathrm{O}(k_0^4).$$

This result is obtained by the uniform theory due to Chester et al. [5], used earlier in our discussion of the Kelvin Pattern. This approximation is uniformly valid and joins up smoothly with the steepest-descents expansion at a distance from the wavefront. (Here again there are nonuniform versions of the theory, involving various cubic approximations to the phase function and valid only in a narrow zone near the wavefront. See our earlier discussion of the Kelvin pattern.) In three dimensions we may expect some similar behaviour, and we must therefore look for a method analogous to Chester et al. [5].

In three dimensions we find by standard methods that

$$\phi(r,t) = \int_0^\infty \frac{\sin\omega t}{\omega} J_0(kr)\, k\, \mathrm{d}k = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin\omega t}{\omega} H_0^{(2)}(kr)\, k\, \mathrm{d}k, \tag{6.1}$$

where

$$\omega = (k\tanh k)^{1/2}$$

is regular near $k = 0$, and where in the second integral the path of integration passes below $k = 0$. Here the trick was, to convert the integral into a double integral, with the result (for the impulsive elevation)

$$\phi(r,t) \sim \frac{2^{1/3}}{4\pi(x/t)^{1/2}} \int_{-\infty}^\infty \mathrm{d}\xi \int_{-\infty}^\infty \mathrm{d}\eta\, G^*(\xi,\eta)\exp(\mathrm{i}t\Psi), \tag{6.2}$$

where

$$\Psi = \Psi(\xi,\eta; -2^{-1/3}\epsilon) = \frac{1}{3}(\xi^3 + \eta^3) - 2^{-1/3}\epsilon(\xi + \eta). \tag{6.3}$$

Following [15] we were then able to construct a Bleistein sequence and derive uniform approximations. The leading approximation for the velocity potential is

$$\phi(r,t) \sim \frac{2^{1/3}\pi}{x^{1/2}t^{1/6}}(E_0\mathrm{Ai}^2 - C_0 t^{-2/3}\mathrm{Ai}'^2), \tag{6.4}$$

where the argument of the Airy function and its derivative is $-2^{-1/3}t^{2/3}\epsilon$. (We recall that the variables are normalized so that $g = 1, h = 1$.) The treatment is in many ways analogous to the treatment of the Kelvin wave pattern but involves two complex variables. For a detailed discussion and numerical results the reader is referred to [14].

7 Conclusion

We have now examined the analytical solution of three problems in the linear theory of water waves. In each case we found that our arguments involved mathematical considerations remote from any physical aspects. From my own experience I could add many more water-wave problems to this list. Pure mathematicians are familiar with this kind of paradox. As an undergraduate student I studied the famous Prime-Number Theorem; the standard proof depends on detailed properties of the Riemann zeta function in the complex plane, although clearly the distribution of prime numbers has nothing to do with the complex plane. I am told that there are now alternative proofs with a closer connection to prime numbers, and it may be that in the future there may be solutions of our water-wave problems which will give us closer physical insights, meanwhile I am happy to have the present constructions because they do give us valid solutions.

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